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An improvement of the Nevanlinna–Gundersen theorem

Nguyen Thi Thu Hang [a], Nguyen Huu Kien [b], Tran Van Tan [b,*]

[a] Department of Mathematics, Hai Phong University, 171-Phan Dang Luu street, Kien An, Hai Phong, Viet Nam
[b] Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy street, Cau Giay, Hanoi, Viet Nam

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ABSTRACT

A well-known result of Nevanlinna states that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ and for four distinct values $a_j \in \mathbb{C} \cup \{\infty\}$, if $\nu_{f-a_j} = \nu_{g-a_j}$ for all $1 \leqslant j \leqslant 4$, then g is a Möbius transformation of f. In 1983, Gundersen generalized the result of Nevanlinna to the case where the above condition is replaced by: $\min\{\nu_{f-a_j}, 1\} = \min\{\nu_{g-a_j}, 1\}$ for $j = 1, 2$ and $\nu_{f-a_j} = \nu_{g-a_j}$ for $j = 3, 4$. In this paper, we prove that the theorem of Gundersen remains valid to the case where $\min\{\nu_{f-a_j}, 1\} = \min\{\nu_{g-a_j}, 1\}$ for $j = 1, 2$, and $\min\{\nu_{f-a_j}, 2\} = \min\{\nu_{g-a_j}, 2\}$ for $j = 3, 4$. Furthermore, we work on the case where $\{a_j\}$ are small functions.

1. Introduction

For a meromorphic function f on $\mathbb{C}$ ($f \not\equiv 0, \infty$), we denote by ν_f, μ_f the zero and pole divisors of f. Set $\nu_{f-\infty} := \mu_f = \nu_{\frac{1}{f}}$.

The characteristic function of f is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\|\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\|f(e^{i\theta})\|\,d\theta \quad (r > 1),$$

where $f = (f_0 : f_1)$ is a reduced representation of f and $\|f\| = (|f_0|^2 + |f_1|^2)^{1/2}$.

We say a meromorphic function a is "small" with respect to f if $T_a(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure). Denote by $\mathcal{R}_f$ the set of all meromorphic functions on $\mathbb{C}$ which are small with respect to f. Then $\mathcal{R}_f$ is a field if f is nonconstant.

In 1926, Nevanlinna [9] proved the following theorem.

Theorem 1.1. *Let f and g be two distinct nonconstant meromorphic functions and let a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \cup \{\infty\}$. Assume that $\nu_{f-a_j} = \nu_{g-a_j}$ for all $1 \leqslant j \leqslant 4$. Then two shared values, say a_3 and a_4, are Picard values, the cross ratio $(a_1, a_2, a_3, a_4) = -1$, and g is a linear transformation of f.*

In 1983 and 1987, Gundersen [4,5] obtained the following improvement of Nevanlinna's theorem.

* Corresponding author.
E-mail addresses: thuhanghp2003@gmail.com (N.T.T. Hang), hkiensp@yahoo.com (N.H. Kien), tranvantanhn@yahoo.com (T.V. Tan).

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Theorem 1.2. *Let f and g be two distinct nonconstant meromorphic functions and let a_1, a_2, a_3, a_4 be four distinct values in $\mathbb{C} \cup \{\infty\}$. Assume that $\min\{\nu_{f-a_i}, 1\} = \min\{\nu_{g-a_i,1}\}$ for $i = 1, 2$, and $\nu_{f-a_j} = \nu_{g-a_j}$ (outside a discrete set of counting function regardless multiplicity is equal to $o(T_f(r))$) for $j = 3, 4$. Then $\nu_{f-a_i} = \nu_{g-a_i}$ for all $i \in \{1, \ldots, 4\}$.*

These results also proved for the case where $\{a_j\}$ are small (with respect to f) meromorphic functions (see Li [7], Shirosaki [10], Tan [11], Yao [13]).

In this paper, we strongly generalize the above results to the case where $\min\{\nu_{f-a_i}, 1\} = \min\{\nu_{g-a_i}, 1\}$ for $i = 1, 2$, $\min\{\nu_{f-a_j}, 2\} = \min\{\nu_{g-a_j}, 2\}$ for $j = 3, 4$, and $\{a_j\}$ are small (with respect to f) functions. We notice that in [3], Gundersen gave examples to explain that Theorem 1.1 does not remain valid to the case where $\min\{\nu_{f-a_i}, 1\} = \min\{\nu_{g-a_i}, 1\}$ for all $i \in \{1, \ldots, 4\}$.

We finally remark that Han, Mori and Tohge [6] also obtained several other interesting extensions of Nevanlinna's result mentioned above. However, in their paper all truncation levels of multiplicity are bigger than 1. Therefore, their results are not stronger than the result of Gundersen.

2. Some lemmas and notations

Let ν be a divisor on $\mathbb{C}$ and k, m be nonnegative integers or $+\infty$. We set

$$^{\geqslant m}\nu^{[k]}(z) := 0 \quad \text{if } \nu(z) < m, \qquad {}^{\geqslant m}\nu^{[k]}(z) := \min\bigl\{\nu(z), k\bigr\} \quad \text{if } \nu(z) \geqslant m.$$

The counting function of ν is defined by

$$^{\geqslant m}N^{[k]}(r, \nu) = \int_1^r \frac{n(t)}{t}\,dt \quad (r > 1), \text{ where } n(t) = \sum_{|z| \leqslant t} {}^{\geqslant m}\nu^{[k]}(z).$$

For a nonzero meromorphic function f on $\mathbb{C}$, we define $^{\geqslant m}N_f^{[k]}(r) := {}^{\geqslant m}N^{[k]}(r, \nu_f)$.

For brevity we will omit the character $[k]$ (respectively $\geqslant m$) in the counting function and in the divisor if $k = \infty$ (respectively $m = 0$).

The proximity function of f is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+\bigl|f\bigl(re^{i\theta}\bigr)\bigr|\,d\theta, \quad \text{where } \log^+ x = \max\{\log x, 0\} \text{ for } x \geqslant 0.$$

We state the First and Second Main Theorems of Value Distribution Theory.

First Main Theorem. *Let f and a be two meromorphic functions on $\mathbb{C}$ such that $(f, a) \not\equiv 0$ then*

$$T_f(r) = N_{\frac{1}{f}}(r) + m(r, f) + O(1) \quad \text{and} \quad N_{f-a}(r) \leqslant T_f(r) + T_a(r) + O(1).$$

As usual, by the notation "$\|\ P$" we mean that the assertion P holds for all $r \in [0, \infty)$ excluding a set of finite Lebesgue measure.

Second Main Theorem *(Moving target version). (See [12, Corollary 6.3].) Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ be q distinct functions in $\mathcal{R}_f$. Then, for each $\epsilon > 0$, the following holds*:

$$\|\ (q - 2 - \epsilon) T_f(r) \leqslant \sum_{i=1}^{q} N_{f-a_i}^{[1]}(r).$$

Let f and a be two meromorphic functions on $\mathbb{C}$ such that $f - a \not\equiv 0$. The *defect* $\delta_f^{[1]}(a)$ of f for a is defined by

$$\delta_f^{[1]}(a) := 1 - \overline{\lim} \frac{N_{f-a}^{[1]}(r)}{T_f(r)}.$$

We have $0 \leqslant \delta_f^{[1]}(a) \leqslant 1$. Furthermore, under the same assumption as in the Second Main Theorem we have

$$\sum_{i=1}^{q} \delta_f^{[1]}(a_i) \leqslant 2.$$

Second Main Theorem *(Classical version). (See [2, Theorem 2.13].) Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^n$ and $H_1, \ldots, H_q$ $(q \geqslant n+1)$ hyperplanes in $\mathbb{C}P^n$ in general position. Then*

$$\| \ (q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N^{[n]}_{(f,H_j)}(r) + o\big(T_f(r)\big).$$

The following result was given by Li and Yang [8] for the case where the condition (i) below is replaced by $N^{[1]}_{h_i}(r) + N^{[1]}_{\frac{1}{h_i}}(r) = o(T_{h_1}(r) + T_{h_2}(r))$. However, their proof remains valid to our statement.

Lemma 2.1. *(See [8, Lemma 7].) Let h_1, h_2 be two nonconstant meromorphic functions satisfying the following conditions:*

(i) *For any $\epsilon > 0$,*

$$\| \ N^{[1]}_{h_i}(r) + N^{[1]}_{\frac{1}{h_i}}(r) \leqslant \epsilon\big(T_{h_1}(r) + T_{h_2}(r)\big).$$

(ii) *There exists $\epsilon_1 > 0$ such that*

$$\| \ N^{[1]}_{(h_1=1=h_2)}(r) \geqslant \epsilon_1\big(T_{h_1}(r) + T_{h_2}(r)\big),$$

where $N^{[1]}_{(h_1=1=h_2)}(r)$ denotes the counting function of those common 1-points regardless multiplicity of h_1, h_2.

Then there exist integers p_1 and p_2 $(|p_1| + |p_2| > 0)$, such that $h_1^{p_1} h_2^{p_2} \equiv 0$.

Lemma 2.2. *(See [13, Lemma 1].) Let f be a nonconstant meromorphic function on $\mathbb{C}$ and let a, b be two distinct meromorphic functions in $\mathcal{R}_f \setminus \{\infty\}$. Set*

$$L(f,a,b) := \begin{vmatrix} f & f' & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix}.$$

Then $L(f,a,b) \not\equiv 0$, and

$$m\left(r, \frac{L(f,a,b)f^k}{(f-a)(f-b)}\right) = o\big(T_f(r)\big), \quad \text{for } k = 0, 1.$$

Lemma 2.3. *Let f, g be nonconstant meromorphic functions and let $\alpha_1, \alpha_2, \alpha_3$ be three distinct meromorphic functions in $\mathcal{R}_f \setminus \{\infty\}$. Assume that $\min\{\mu_f, 1\} = \min\{\mu_g, 1\}$ and $\min\{\nu_{f-\alpha_j}, 1\} = \min\{\nu_{g-\alpha_j}, 1\}$, $j = 1, 2, 3$. Set*

$$\Phi = \Phi(\alpha_1, \alpha_2, \alpha_3) := \frac{L(f, \alpha_1, \alpha_2)(f-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)} - \frac{L(g, \alpha_1, \alpha_2)(g-\alpha_3)}{(g-\alpha_1)(g-\alpha_2)}.$$

If $\Phi(\alpha_1, \alpha_2, \alpha_3) \cdot \Phi(\alpha_3, \alpha_2, \alpha_1) \not\equiv 0$, then the following assertions hold:

(i) $N_{\frac{1}{\Phi}}(r) \leqslant \sum_{i=1,2} N^{[1]}(r, |\nu_{f-\alpha_i} - \nu_{g-\alpha_i}|) + N^{[1]}(r, |\mu_f - \mu_g|) + o(T_f(r))$, *where $|\cdot|$ is the absolute value notation.*

(ii) $T_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(r) \leqslant N^{[1]}(r, |\nu_{f-\alpha_2} - \nu_{g-\alpha_2}|) + N^{[1]}(r, |\mu_f - \mu_g|) + o(T_f(r))$.

(iii) $N^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(r) \leqslant 2\sum_{i=1}^{3} N^{[1]}(r, |\nu_{f-\alpha_i} - \nu_{g-\alpha_i}|) + 2N^{[1]}(r, |\mu_f - \mu_g|) + o(T_f(r))$.

(iv) $N^{[1]}_{\frac{f-\alpha_3}{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}}(r) \leqslant 2\sum_{i=1}^{3} N^{[1]}(r, |\nu_{f-\alpha_i} - \nu_{g-\alpha_i}|) + 2N^{[1]}(r, |\mu_f - \mu_g|) + o(T_f(r))$.

Proof. Set

$$\gamma := 2\sum_{1\leqslant i<j\leqslant 3} \nu_{\alpha_i-\alpha_j} + 2\sum_{i=1}^{3}(\nu_{\alpha_i} + \mu_{\alpha_i}).$$

Since $\alpha_i \in \mathcal{R}_f$, we get $N(r,\gamma) = o(T_f(r))$. For an arbitrary point z_0, set

$$k(z_0) := \mu_{\frac{L(f,\alpha_1,\alpha_2)(f-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)}}(z_0) \quad \text{and} \quad t(z_0) := \mu_{\frac{L(g,\alpha_1,\alpha_2)(g-\alpha_3)}{(g-\alpha_1)(g-\alpha_2)}}(z_0).$$

We now prove that

$$k(z_0) \leqslant \max\{\nu^{[1]}_{f-\alpha_1}(z_0), \nu^{[1]}_{f-\alpha_2}(z_0)\} + \mu^{[1]}_f(z_0) + \gamma(z_0). \tag{2.1}$$

Without loss of generality we may assume that $\nu_{f-\alpha_1}(z_0) \geqslant \nu_{f-\alpha_2}(z_0)$. Then

$$\nu_{\alpha_2-\alpha_1}(z_0) = \nu_{(f-\alpha_1)-(f-\alpha_2)}(z_0) \geqslant \nu_{f-\alpha_2}(z_0). \tag{2.2}$$

On the other hand, by an easy computation we get that

$$\frac{L(f,\alpha_1,\alpha_2)(f-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)} = \left((\alpha_1-\alpha_2)' - \frac{(f-\alpha_1)'}{f-\alpha_1}(\alpha_1-\alpha_2)\right)\left(1+\frac{\alpha_2-\alpha_3}{f-\alpha_2}\right). \tag{2.3}$$

Therefore, by (2.2)

$$\begin{aligned}
k(z_0) &\leqslant \nu^{[1]}_{f-\alpha_1}(z_0) + \mu^{[1]}_{f-\alpha_1}(z_0) + \mu_{(\alpha_1-\alpha_2)'}(z_0) + \mu_{\alpha_2-\alpha_3}(z_0) + \nu_{f-\alpha_2}(z_0)\\
&\leqslant \nu^{[1]}_{f-\alpha_1}(z_0) + \mu^{[1]}_f(z_0) + \mu_{\alpha_1}(z_0) + \mu_{\alpha_1-\alpha_2}(z_0) + 2\mu_{\alpha_2-\alpha_3}(z_0) + \nu_{\alpha_2-\alpha_1}(z_0)\\
&\leqslant \nu^{[1]}_{f-\alpha_1}(z_0) + \mu^{[1]}_f(z_0) + \gamma(z_0)\\
&\leqslant \max\{\nu^{[1]}_{f-\alpha_1}(z_0), \nu^{[1]}_{f-\alpha_2}(z_0)\} + \mu^{[1]}_f(z_0) + \gamma(z_0).
\end{aligned}$$

Thus, we get (2.1). Similarly, we have

$$t(z_0) \leqslant \max\{\nu^{[1]}_{g-\alpha_1}(z_0), \nu^{[1]}_{g-\alpha_2}(z_0)\} + \mu^{[1]}_g(z_0) + \gamma(z_0). \tag{2.4}$$

For the proof of (i), it suffices to show that

$$\begin{aligned}
\mu_\Phi(z_0) \leqslant{}& \min\{|\nu_{f-\alpha_1}(z_0) - \nu_{g-\alpha_1}(z_0)|, 1\} + \min\{|\nu_{f-\alpha_2}(z_0) - \nu_{g-\alpha_2}(z_0)|, 1\}\\
&+ \min\{|\mu_f(z_0) - \mu_g(z_0)|, 1\} + 4\gamma(z_0).
\end{aligned} \tag{2.5}$$

Case 1. $\gamma(z_0) \geqslant 1$.

By (2.1), we get

$$k(z_0) \leqslant \nu^{[1]}_{f-\alpha_2}(z_0) + \nu^{[1]}_{f-\alpha_1}(z_0) + \mu^{[1]}_f(z_0) + \gamma(z_0) \leqslant 3 + \gamma(z_0) \leqslant 4\gamma(z_0).$$

Similarly, by (2.4), we get

$$t(z_0) \leqslant 4\gamma(z_0).$$

Therefore, $\mu_\Phi(z_0) = \max\{k(z_0), t(z_0)\} \leqslant 4\gamma(z_0)$. Then we get (2.5).

Case 2. $\gamma(z_0) = 0$ and there exists $i \in \{1,2\}$ such that $\nu_{f-\alpha_i}(z_0) = \nu_{g-\alpha_i}(z_0) := m \geqslant 1$, say $i = 1$.

Then, it is easy to see that

$$f(z_0) - \alpha_i(z_0) \neq 0 \quad \text{and} \quad g(z_0) - \alpha_i(z_0) \neq 0, \quad \text{for } i = 2, 3. \tag{2.6}$$

On a neighborhood of z_0, there exist holomorphic functions h, u such that $h(z_0), u(z_0) \neq 0$ and $f - \alpha_1 = (z-z_0)^m h$, $g - \alpha_1 = (z-z_0)^m u$. By (2.3), we have

$$\frac{L(f,\alpha_1,\alpha_2)(f-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)} = \left((\alpha_1-\alpha_2)' - \left(\frac{m}{z-z_0} + \frac{h'}{h}\right)(\alpha_1-\alpha_2)\right)\left(1+\frac{\alpha_2-\alpha_3}{f-\alpha_2}\right). \tag{2.7}$$

Similarly,

$$\frac{L(g,\alpha_1,\alpha_2)(g-\alpha_3)}{(g-\alpha_1)(g-\alpha_2)} = \left((\alpha_1-\alpha_2)' - \left(\frac{m}{z-z_0} + \frac{u'}{u}\right)(\alpha_1-\alpha_2)\right)\left(1+\frac{\alpha_2-\alpha_3}{g-\alpha_2}\right). \tag{2.8}$$

On the other hand, $f(z_0) - \alpha_2(z_0) = \alpha_1(z_0) - \alpha_2(z_0) = g(z_0) - \alpha_2(z_0) \neq 0, \infty$. Hence, by (2.7) and (2.8), we have that Φ is holomorphic at z_0. Hence, we get (2.5).

Case 3. $\gamma(z_0) = 0$ and $\mu_f(z_0) = \mu_g(z_0) = n \geqslant 1$.

On a neighborhood of z_0, we have $f = (z-z_0)^{-n} v(z)$, $g = (z-z_0)^{-n} w(z)$ where $v(z), w(z)$ are holomorphic functions and $v(z_0), w(z_0) \neq 0$. Then by an easy computation, on a neighborhood of z_0 we have

$$\frac{L(f,\alpha_1,\alpha_2)(f-\alpha_3)}{(f-\alpha_1)(f-\alpha_2)}=\frac{1}{z-z_0}\cdot\frac{n(\alpha_1-\alpha_2)}{1-\alpha_1\frac{(z-z_0)^n}{v}}\cdot\frac{1-\alpha_3(z-z_0)^n v^{-1}}{1-\alpha_2(z-z_0)^n v^{-1}}+A(z)\quad\text{and}$$

$$\frac{L(g,\alpha_1,\alpha_2)(g-\alpha_3)}{(g-\alpha_1)(g-\alpha_2)}=\frac{1}{z-z_0}\cdot\frac{n(\alpha_1-\alpha_2)}{1-\alpha_1\frac{(z-z_0)^n}{w}}\cdot\frac{1-\alpha_3(z-z_0)^n w^{-1}}{1-\alpha_2(z-z_0)^n w^{-1}}+B(z)$$

where $A(z), B(z)$ are holomorphic functions. Therefore, Φ is holomorphic at z_0 (note that $\gamma(z_0)=0$). Then, we get (2.5).

Case 4. $\gamma(z_0)=0$ and $0=\nu_{f-\alpha_1}(z_0)=\nu_{f-\alpha_2}(z_0)=\mu_f(z_0)$.
Then by (2.1) and (2.4), we have $k(z_0)=0=t(z_0)$. Therefore, $\mu_\Phi(z_0)=\max\{k(z_0),t(z_0)\}=0$ and then we get (2.5).

Case 5. $0=\gamma(z_0)=\nu_{f-\alpha_1}(z_0)=\nu_{f-\alpha_2}(z_0)$, and $\mu_f(z_0)\neq\mu_g(z_0)$.
Then by (2.1) and (2.4), we have $k(z_0)\leqslant\mu_f^{[1]}(z_0)\leqslant 1$ and $t(z_0)\leqslant\mu_g^{[1]}(z_0)\leqslant 1$. Therefore, $\mu_\Phi(z_0)=\max\{k(z_0),t(z_0)\}\leqslant 1=\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}$. This implies (2.5).

We now check (2.5) in the last case.

Case 6. $\gamma(z_0)=0$, $\mu_f(z_0)\neq\mu_g(z_0)$, and $\nu_{f-\alpha_1}(z_0)\neq\nu_{g-\alpha_1}(z_0)$ or $\nu_{f-\alpha_2}(z_0)\neq\nu_{g-\alpha_2}(z_0)$.
Then we have $\min\{|\nu_{f-\alpha_1}(z_0)-\nu_{g-\alpha_1}(z_0)|,1\}=1$ or $\min\{|\nu_{f-\alpha_2}(z_0)-\nu_{g-\alpha_2}(z_0)|,1\}=1$, and $\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}=1$. On the other hand by (2.1) and (2.4), we have $k(z_0),t(z_0)\leqslant 2$. Hence, we get

$$\begin{aligned}\mu_\Phi(z_0)&=\max\{k(z_0),t(z_0)\}\leqslant 2\\&\leqslant\min\{|\nu_{f-\alpha_1}(z_0)-\nu_{g-\alpha_1}(z_0)|,1\}+\min\{|\nu_{f-\alpha_2}(z_0)-\nu_{g-\alpha_2}(z_0)|,1\}\\&\quad+\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}+4\gamma(z_0).\end{aligned}$$

Then we get (2.5).

The proof of (2.5) has been completed, then we get assertion (i).

We now prove assertion (ii).

Let z_0 be an arbitrary point, by (2.5), we have

$$\begin{aligned}\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)}(z_0)&\leqslant\min\{|\nu_{f-\alpha_1}(z_0)-\nu_{g-\alpha_1}(z_0)|,1\}+\min\{|\nu_{f-\alpha_2}(z_0)-\nu_{g-\alpha_2}(z_0)|,1\}\\&\quad+\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}+4\gamma(z_0).\end{aligned}\tag{2.9}$$

Similarly,

$$\begin{aligned}\mu_{\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0)&\leqslant\min\{|\nu_{f-\alpha_3}(z_0)-\nu_{g-\alpha_3}(z_0)|,1\}+\min\{|\nu_{f-\alpha_2}(z_0)-\nu_{g-\alpha_2}(z_0)|,1\}\\&\quad+\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}+4\gamma(z_0).\end{aligned}\tag{2.10}$$

If $\gamma(z_0)\neq 0$, then by (2.9), (2.10) we have

$$\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2\alpha_1)}(z_0)\leqslant 14\gamma(z_0).\tag{2.11}$$

Noting that if $\gamma(z_0)=0$, $\nu_{f-\alpha_3}(z_0)\neq 0$, then z_0 is a zero point of $\Phi(\alpha_1,\alpha_2,\alpha_3)$, and if $\gamma(z_0)=0$, $\nu_{f-\alpha_1}(z_0)\neq 0$, then z_0 is a zero point of $\Phi(\alpha_3,\alpha_2,\alpha_1)$. Therefore, from (2.9), (2.10) we get that if $\gamma(z_0)=0$, then

$$\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0)\leqslant\min\{|\nu_{f-\alpha_2}(z_0)-\nu_{g-\alpha_2}(z_0)|,1\}+\min\{|\mu_f(z_0)-\mu_g(z_0)|,1\}.\tag{2.12}$$

From (2.11) and (2.12) we have

$$\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}\leqslant\min\{|\nu_{f-\alpha_2}-\nu_{g-\alpha_2}|,1\}+\min\{|\mu_f-\mu_g|,1\}+14\gamma.$$

Therefore,

$$N(r,\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)})\leqslant N^{[1]}(r,|\nu_{f-\alpha_2}-\nu_{g-\alpha_2}|)+N^{[1]}(r,|\mu_f-\mu_g|)+o(T_f(r)).$$

On the other hand, by Lemma 2.2, we have $m(r,\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1))=o(T_f(r))$. Therefore, by the First Main Theorem we have

$$\begin{aligned}T_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(r)&=N(r,\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)})+m(r,\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1))\\&\leqslant N^{[1]}(r,|\nu_{f-\alpha_2}-\nu_{g-\alpha_2}|)+N^{[1]}(r,|\mu_f-\mu_g|)+o(T_f(r)).\end{aligned}$$

We get assertion (ii).

We finally prove assertions (iii) and (iv).

Let z_0 be an arbitrary point satisfying $\gamma(z_0)=0$.

– If $\mu_f(z_0)=\mu_g(z_0)\geqslant 1$ then $\nu_{\frac{g-\alpha_1}{f-\alpha_3}}(z_0)=0=\mu_{\frac{g-\alpha_1}{f-\alpha_3}}(z_0)$ and by an argument similar to Case 3, we have that $\Phi(\alpha_1,\phi_2,\phi_3)$ and $\Phi(\alpha_3,\phi_2,\phi_1)$ are holomorphic at z_0. Therefore

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)\leqslant \nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0),\quad \text{and} \tag{2.13}$$

$$\mu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=0. \tag{2.14}$$

– If $\nu_{f-\alpha_3}(z_0)=\nu_{g-\alpha_3}(z_0)\geqslant 1$ then $\Phi(\alpha_1,\alpha_2,\alpha_3)(z_0)=0$ and by an argument similar to Case 2, we have that $\Phi(\alpha_3,\alpha_2,\alpha_1)$ is holomorphic at z_0. On the other hand, since $\nu_{g-\alpha_3}(z_0)\geqslant 1$ and $\gamma(z_0)=0$, we have $(g-\alpha_1)(z_0)\neq 0,\infty$. Therefore, we have

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=1=\nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0),\quad \text{and} \tag{2.15}$$

$$\mu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=0. \tag{2.16}$$

– If $\nu_{f-\alpha_1}(z_0)=\nu_{g-\alpha_1}(z_0)\geqslant 1$, then $\Phi(\alpha_3,\alpha_2,\alpha_1)(z_0)=0$ and by Case 2, we have that $\Phi(\alpha_1,\alpha_2,\alpha_3)$ is holomorphic at z_0. Therefore

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=1=\nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0). \tag{2.17}$$

– If $\nu_{f-\alpha_2}(z_0)=\nu_{g-\alpha_2}(z_0)\geqslant 1$, then $(f-\alpha_3)(z_0)\neq 0$, $(g-\alpha_1)(z_0)\neq 0$ and by an argument similar to Case 2, we have that $\Phi(\alpha_1,\alpha_2,\alpha_3)$ and $\Phi(\alpha_3,\alpha_2,\alpha_1)$ are holomorphic at z_0. Then

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)\leqslant \nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0),\quad \text{and} \tag{2.18}$$

$$\mu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=0. \tag{2.19}$$

– If $\nu_{\Phi(\alpha_1,\alpha_2,\alpha_3)}(z_0)\geqslant 1$, and $\mu_f(z_0)=\nu_{f-\alpha_2}(z_0)=\nu_{f-\alpha_3}(z_0)=0$, then $\Phi(\alpha_3,\alpha_2,\alpha_1)$ is holomorphic at z_0. Then

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=1=\nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0). \tag{2.20}$$

From (2.13), (2.15), (2.17), (2.18), and (2.20) we have

$$\nu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}\leqslant \nu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}+\min\bigl\{|\mu_f-\mu_g|,1\bigr\}+\sum_{i=1}^{3}\min\bigl\{|\nu_{f-\alpha_i}-\nu_{g-\alpha_i}|,1\bigr\}+\gamma.$$

Therefore, by the First Main Theorem and by assertion (ii) we have

$$\begin{aligned}
N^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(r) &\leqslant N^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(r)+N^{[1]}\bigl(r,|\mu_f-\mu_g|\bigr)+\sum_{i=1}^{3}N^{[1]}\bigl(r,|\nu_{f-\alpha_i}-\nu_{g-\alpha_i}|\bigr)+N_\gamma(r)\\
&\leqslant T_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(r)+\sum_{i=1}^{3}N^{[1]}\bigl(r,|\nu_{f-\alpha_i}-\nu_{g-\alpha_i}|\bigr)+N^{[1]}\bigl(r,|\mu_f-\mu_g|\bigr)+o\bigl(T_f(r)\bigr)\\
&\leqslant 2\sum_{i=1}^{3}N^{[1]}\bigl(r,|\nu_{f-\alpha_i}-\nu_{g-\alpha_i}|\bigr)+2N^{[1]}\bigl(r,|\mu_f-\mu_g|\bigr)+o\bigl(T_f(r)\bigr).
\end{aligned}$$

Then assertion (iii) holds.

– If $\mu_{\Phi(\alpha_1,\alpha_2,\alpha_3)}(z_0)\geqslant 1$, and $\mu_f(z_0)=\nu_{f-a_2}(z_0)=\nu_{f-a_3}(z_0)=0$, then $\Phi(\alpha_3,\alpha_2,\alpha_1)$ is holomorphic at z_0. Then

$$\mu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}(z_0)=1=\mu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}(z_0). \tag{2.21}$$

From (2.14), (2.16), (2.19), (2.21), we have

$$\mu^{[1]}_{\frac{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}{f-\alpha_3}}\leqslant \mu^{[1]}_{\Phi(\alpha_1,\alpha_2,\alpha_3)\cdot\Phi(\alpha_3,\alpha_2,\alpha_1)}+\min\bigl\{|\mu_f-\mu_g|,1\bigr\}+\sum_{i=2,3}\min\bigl\{|\nu_{f-\alpha_i}-\nu_{g-\alpha_i}|,1\bigr\}+\gamma.$$

Therefore, by the First Main Theorem and by assertion (ii) again we have

$$N^{[1]}_{\frac{f-\alpha_3}{\Phi(\alpha_1,\alpha_2,\alpha_3)(g-\alpha_1)}}(r) \leqslant 2\sum_{i=1}^{3} N^{[1]}\big(r, |\nu_{f-\alpha_i} - \nu_{g-\alpha_i}|\big) + 2N^{[1]}\big(r, |\mu_f - \mu_g|\big) + o\big(T_f(r)\big). \tag{2.22}$$

Therefore, assertion (iv) holds. □

Let $\mathcal{G}$ be a torsion free abelian group and $A = (x_1, \ldots, x_q)$ be a q-tuple of elements x_i in $\mathcal{G}$. Let $1 < s < r \leqslant q$. We say that A has the property $P_{r,s}$ if any r elements $x_{p_1}, \ldots, x_{p_r}$ in A satisfy the condition that for any subset $I \subset \{p_1, \ldots, p_r\}$ with $\#I = s$, there exists a subset $J \subset \{p_1, \ldots, p_r\}$, $J \neq I$, $\#J = s$ such that $\prod_{i\in I} x_i = \prod_{j\in J} x_j$.

Lemma 2.4. *(See [1, Lemma 5.1].) If A has the property $P_{r,s}$, then there exists a subset $\{i_1, \ldots, i_{q-r+2}\}$ of $\{1, \ldots, q\}$ such that $x_{i_1} = \cdots = x_{i_{q-r+2}}$.*

3. Uniqueness theorem

Let f be a nonconstant meromorphic function on $\mathbb{C}$. Denote by $\mathcal{A}_f$ the set of all functions $a \in \mathcal{R}_f$ such that $\delta_f^{[1]}(a) = 1$. It is clear that $\#\mathcal{A}_f \leqslant 2$.

Theorem 3.1. *Let f and g be two distinct nonconstant meromorphic functions on $\mathbb{C}$ with reduced presentations $f = (f_0 : f_1)$, $g = (g_0 : g_1)$. Let a_j $(1 \leqslant j \leqslant 4)$ be four distinct functions in $\mathcal{R}_f$ with reduced presentations $a_j = (a_{j0} : a_{j1})$. Assume that $\min\{\nu_{f-a_j}, 1\} = \min\{\nu_{g-a_j}, 1\}$ for $j = 1, 2$, and $\min\{\nu_{f-a_i}, 2\} = \min\{\nu_{g-a_i}, 2\}$ for $i = 3, 4$. Then the following assertions hold:*

(i) *For any $\epsilon > 0$,*

$$\| \; N(r, D_{(f,g,a_j)}) \leqslant \epsilon\big(T_f(r) + T_g(r)\big), \quad j \in \{1, \ldots, 4\}$$

where $D_{(f,g,a_j)} = 0$ on $\{z\colon \nu_{f-a_j}(z) = \nu_{g-a_j}(z)\}$ and $D_{(f,g,a_j)} = 1$ on $\{z\colon \nu_{f-a_j}(z) \neq \nu_{g-a_j}(z)\}$.

(ii) *The set $\{a_1, a_2, a_3, a_4\} \cap \mathcal{A}_f$ contains exactly 2 elements, say a_3, a_4 and the cross ratio (a_1, a_2, a_3, a_4) is identically equal to -1.*

(iii) *$g \equiv S(f)$, $S(a_1) \equiv a_1$, $S(a_2) \equiv a_2$, $S(a_3) \equiv a_4$, and $S(a_4) \equiv a_3$, where*

$$S := \begin{pmatrix} -a_{11} & a_{10} \\ a_{21} & -a_{20} \end{pmatrix}^{-1} \circ \begin{pmatrix} a_{11} & -a_{10} \\ a_{21} & -a_{20} \end{pmatrix}.$$

Proof. We first remark that by the assumption of Theorem 3.1 we have $\mathrm{Zero}(f - a_j) = \mathrm{Zero}(g - a_j)$ for all $1 \leqslant j \leqslant 4$. Therefore, by the First and Second Main Theorems we easily get that $T_f(r) = O(T_g(r))$. In particular, $\mathcal{R}_f = \mathcal{R}_g$.

For the proof of (i), we may assume that a_1, a_2 are constants, and $a_4 \equiv \infty$. Set $S_0 := (\bigcup_{1\leqslant i\neq j\leqslant 3} \mathrm{Zero}(a_i - a_j)) \cup (\bigcup_{i=1}^{3} \mathrm{Pole}(a_i))$.

For each pair of positive integers (k, l) we denote $N^{[1]}_{(a_j;k,l)}(r)$ the counting function (regardless multiplicity) of all points z satisfying $\nu_{f-a_j}(z) = k$, and $\nu_{g-a_j}(z) = l$.

Since $\min\{\nu_{f-a_i}, 1\} = \min\{\nu_{f-a_i}, 1\}$ $(i = 1, 2)$ and $\min\{\nu_{f-a_j}, 2\} = \min\{\nu_{f-a_j}, 2\}$ $(j = 3, 4)$, we have

$$\begin{aligned} &N^{[1]}_{f-a_1}(r) + \sum_{k,l\geqslant 2} N^{[1]}_{(a_1;k,l)}(r) + N^{[1]}_{f-a_2}(r) + \sum_{k,l\geqslant 2} N^{[1]}_{(a_2;k,l)}(r) + N^{[2]}_{f-a_3}(r) + N^{[2]}_{f-a_4}(r) \\ &\quad \leqslant N_{f-g}(r) \leqslant T_{f-g}(r) + o\big(T_f(r)\big) \leqslant T_f(r) + T_g(r) + o\big(T_f(r)\big). \end{aligned} \tag{3.1}$$

Similarly,

$$\begin{aligned} &N^{[1]}_{g-a_1}(r) + \sum_{k,l\geqslant 2} N^{[1]}_{(a_1;k,l)}(r) + N^{[1]}_{g-a_2}(r) + \sum_{k,l\geqslant 2} N^{[1]}_{(a_2;k,l)}(r) + N^{[2]}_{g-a_3}(r) + N^{[2]}_{g-a_4}(r) \\ &\quad \leqslant N_{f-g}(r) \leqslant T_{f-g}(r) + o\big(T_f(r)\big) \leqslant T_f(r) + T_g(r) + o\big(T_f(r)\big). \end{aligned} \tag{3.2}$$

If a_3 is constant, then by the Second Main Theorem (for fixed distinct points $a_1, \ldots, a_4$) we have

$$\sum_{i=1}^{4} \big(N^{[1]}_{f-a_i}(r) + N^{[1]}_{g-a_i}(r)\big) \geqslant 2\big(T_f(r) + T_g(r)\big) - o\big(T_f(r)\big).$$

Combining with (3.1), (3.2) we have

$$\sum_{i=3,4} \big(N^{[2]}_{f-a_j}(r) - N^{[1]}_{f-a_j}(r)\big) + \sum_{i=3,4} \big(N^{[2]}_{g-a_j}(r) - N^{[1]}_{g-a_j}(r)\big) = o\big(T_f(r)\big).$$

This implies that

$$ {}^{\geqslant 2}N^{[1]}_{f-a_j}(r) + {}^{\geqslant 2}N^{[1]}_{g-a_j}(r) = o\bigl(T_f(r)\bigr), \quad j = 3, 4. \tag{3.3}$$

Set $\Omega_j = \{z\colon\ \nu_{f-a_j}(z) \geqslant 2 \text{ or } \nu_{g-a_j}(z) \geqslant 2\}$, $j = 3, 4$. Then by (3.3) we have $N^{[1]}_{\Omega_j}(r) = o(T_f(r))$. On the other hand, $\min\{\nu_{f-a_i}, 1\} = \min\{\nu_{g-a_i}, 1\}$, $i = 1, 2$, and $\nu_{f-a_j} = \nu_{g-a_j}$ on $\mathbb{C} \setminus \Omega_j$, $j = 3, 4$. Hence, by Theorem 1.2 assertion (i) holds in this case.

We now prove (i) for the case where a_3 is nonconstant (and $a_1, a_2 \in \mathbb{C}$, $a_4 = \infty$).

By the Second Main Theorem, for any $\epsilon > 0$, we have

$$\Bigl\| \left(2 - \frac{\epsilon}{4}\right)\bigl(T_f(r) + T_g(r)\bigr) \leqslant \sum_{i=1}^{4} N^{[1]}_{f-a_i}(r) + \sum_{i=1}^{4} N^{[1]}_{g-a_i}(r).$$

Combining with (3.1) and (3.2), we get

$$\begin{aligned}\Bigl\| \ &\sum_{k,l \geqslant 2} N^{[1]}_{(a_1;k,l)}(r) + \sum_{k,l \geqslant 2} N^{[1]}_{(a_2;k,l)}(r) + \sum_{i=3,4}\bigl(N^{[1]}_{f-a_i}(r) - N^{[2]}_{f-a_i}(r)\bigr) + \sum_{i=3,4}\bigl(N^{[1]}_{g-a_i}(r) - N^{[2]}_{g-a_i}(r)\bigr)\\ &\leqslant \frac{\epsilon}{3}\bigl(T_f(r) + T_g(r)\bigr).\end{aligned} \tag{3.4}$$

This implies that

$$\Bigl\| \sum_{k,l \geqslant 2} N^{[1]}_{(a_j;k,l)}(r) \leqslant \frac{\epsilon}{3}\bigl(T_f(r) + T_g(r)\bigr) \quad (j = 1, 2) \quad \text{and} \tag{3.5}$$

$$\Bigl\| \sum_{i=3,4}\bigl({}^{\geqslant 2}N^{[1]}_{f-a_i}(r) + {}^{\geqslant 2}N^{[1]}_{g-a_i}(r)\bigr) \leqslant \frac{\epsilon}{3}\bigl(T_f(r) + T_g(r)\bigr). \tag{3.6}$$

Therefore, for $i = 3, 4$

$$\| \ N(r, D_{(f,g,a_i)}) \leqslant {}^{\geqslant 2}N^{[1]}_{f-a_i}(r) + {}^{\geqslant 2}N^{[1]}_{g-a_i}(r) \leqslant \frac{\epsilon}{3}\bigl(T_f(r) + T_g(r)\bigr) \tag{3.7}$$

(note that $D_{(f,g,a_i)}(z) = 0$ if $\nu_{f-a_i}(z) = \nu_{g-a_i}(z)$ and $D_{(f,g,a_i)}(z) = 1$ if $\nu_{f-a_i}(z) \neq \nu_{g-a_i}(z)$).

Set

$$\begin{aligned}\Phi_1 &:= \Phi(a_1, a_3, a_2) = \frac{L(f, a_1, a_3)(f - a_2)}{(f - a_1)(f - a_3)} - \frac{L(g, a_1, a_3)(g - a_2)}{(g - a_1)(g - a_3)}, \quad \text{and}\\ \Phi_2 &:= \Phi(a_2, a_3, a_1) = \frac{L(f, a_2, a_3)(f - a_1)}{(f - a_2)(f - a_3)} - \frac{L(g, a_2, a_3)(g - a_1)}{(g - a_2)(g - a_3)}.\end{aligned}$$

Consider two cases.

Case 1. $\Phi_1 \cdot \Phi_2 \equiv 0$. Without loss of generality, we may assume that $\Phi_2 \equiv 0$.

Let $z_0 \in \mathrm{Zero}(f - a_2) \setminus S_0$ be an arbitrary point. Set $k := \nu_{f-a_2}(z_0) \geqslant 1$, $t := \nu_{g-a_2}(z_0) \geqslant 1$. On a neighborhood of z_0, we have $f(z) - a_2(z) = (z - z_0)^k h(z)$, $g(z) - a_2(z) = (z - z_0)^t u(z)$, where $h(z), u(z)$ are holomorphic functions with $h(z_0), u(z_0) \neq 0$.

By an easy computation, we get

$$\frac{L(f, a_2, a_3)(f - a_1)}{(f - a_2)(f - a_3)} = \left((a_2 - a_3)' - \frac{(f - a_2)'}{f - a_2}(a_2 - a_3)\right)\left(1 + \frac{a_3 - a_1}{f - a_3}\right).$$

Therefore, on a neighborhood of z_0 we have

$$\frac{L(f, a_2, a_3)(f - a_1)}{(f - a_2)(f - a_3)} = \left((a_2 - a_3)' - \left(\frac{k}{z - z_0} + \frac{h'(z)}{h(z)}\right)(a_2 - a_3)\right)\left(1 + \frac{a_3 - a_1}{f - a_3}\right). \tag{3.8}$$

Similarly,

$$\frac{L(g, a_2, a_3)(g - a_1)}{(g - a_2)(g - a_3)} = \left((a_2 - a_3)' - \left(\frac{t}{z - z_0} + \frac{u'(z)}{u(z)}\right)(a_2 - a_3)\right)\left(1 + \frac{a_3 - a_1}{g - a_3}\right). \tag{3.9}$$

On the other hand, $f(z_0) = a_2(z_0) = g(z_0)$, and $a_i(z_0) - a_j(z_0) \neq 0, \infty$ for all $1 \leqslant i \neq j \leqslant 3$. Hence, from (3.8), (3.9) and $\Phi_2 \equiv 0$, we get $k = t$. Thus, we have

$$\nu_{f-a_2} = \nu_{g-a_2} \quad \text{on } \mathbb{C} \setminus S_0.$$

This implies that

$$N(r, D_{(f,g,a_2)}) = o\big(T_f(r)\big). \tag{3.10}$$

We now prove that for any $\epsilon > 0$,

$$\| \ N(r, D_{(f,g,a_1)}) \leqslant \epsilon T_f(r). \tag{3.11}$$

Set

$$\psi_1 := (a_2 - a_1)\left(\frac{(f-a_2)'}{f-a_2} - \frac{(g-a_2)'}{g-a_2}\right) \quad \text{and} \quad \psi_2 := (a_3 - a_1)\left(\frac{(f-a_3)'}{f-a_3} - \frac{(g-a_3)'}{g-a_3}\right).$$

If $\psi_1 \equiv 0$, then $(f-a_2)'(g-a_2) - (f-a_2)(g-a_2)' \equiv 0$. Therefore, $c := \frac{f-a_2}{g-a_2}$ is constant. Since $f \not\equiv g$, we have $c \neq 1$. We have $f - a_1 = c(g-a_1) + (1-c)(a_2 - a_1)$. On the other hand, $\mathrm{Zero}(f-a_1) = \mathrm{Zero}(g-a_1)$. Hence, $\mathrm{Zero}(f-a_1) = \mathrm{Zero}(a_2 - a_1)$. This implies that

$$N(r, D_{(f,g,a_1)}) \leqslant N^{[1]}_{f-a_1}(r) \leqslant N^{[1]}_{a_2-a_1}(r) = o\big(T(r)\big).$$

Similarly, if $\psi_2 \equiv 0$ then

$$N(r, D_{(f,g,a_1)}) \leqslant N^{[1]}_{f-a_1}(r) \leqslant N^{[1]}_{a_3-a_1}(r) = o\big(T(r)\big).$$

Therefore, if $\psi_1 \cdot \psi_2 \equiv 0$ we get (3.11).

Next we check (3.11) for the case $\psi_1 \cdot \psi_2 \not\equiv 0$.

By an easy computation we have

$$\psi := \psi_1 - \psi_2 = \left(\frac{(f-a_2)'}{f-a_2} - \frac{(f-a_3)'}{f-a_3}\right)(f-a_1) - \left(\frac{(g-a_2)'}{g-a_2} - \frac{(g-a_3)'}{g-a_3}\right)(g-a_1). \tag{3.12}$$

Subcase 1a. $\psi \not\equiv 0$.

By the Lemma of Logarithmic Derivative, we have

$$m(r,\psi) \leqslant m(r,\psi_1) + m(r,\psi_2) = o\big(T_f(r)\big).$$

Therefore, by the First Main Theorem we have

$$T_\psi(r) \leqslant T_{\psi_1}(r) + T_{\psi_2}(r) + O(1) \leqslant N(r, \mu_{\psi_1}) + N(r, \mu_{\psi_2}) + o\big(T_f(r)\big). \tag{3.13}$$

It is easy to see that $\mu_{\psi_i} \leqslant 1 + \sum_{j=1}^{3} \mu_{a_j}$ and ψ_1 (respectively ψ_2) is holomorphic at any point $z_0 \in \{z\colon\ \nu_f(z) = \nu_g(z)$ or $\nu_{f-a_2}(z) = \nu_{g-a_2}(z)\} \setminus S_0$ (respectively $z_0 \in \{z\colon\ \nu_f(z) = \nu_g(z)$ or $\nu_{f-a_3}(z) = \nu_{g-a_3}(z)\} \setminus S_0$). Therefore, we have

$$\begin{aligned} N(r,\mu_{\psi_1}) &\leqslant N^{[1]}(r,\mu_{\psi_1}) + o\big(T_f(r)\big) \\ &\leqslant N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + N^{[1]}\big(r, |\mu_f - \mu_g|\big) + o\big(T_f(r)\big) \\ &= N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) + o\big(T_f(r)\big) \\ &\leqslant N(r, D_{(f,g,a_2)}) + N(r, D_{(f,g,a_4)}) + o\big(T_f(r)\big) \end{aligned}$$

and

$$\begin{aligned} N(r,\mu_{\psi_2}) &\leqslant N^{[1]}(r,\mu_{\psi_2}) + o\big(T_f(r)\big) \\ &\leqslant N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\mu_f - \mu_g|\big) + o\big(T_f(r)\big) \\ &= N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) + o\big(T_f(r)\big) \\ &\leqslant N(r, D_{(f,g,a_3)}) + N(r, D_{(f,g,a_4)}) + o\big(T_f(r)\big). \end{aligned}$$

Combining with (3.7) and (3.10), for any $\epsilon > 0$ we have

$$\| \ N(r,\mu_{\psi_1}) \leqslant \frac{\epsilon}{3} T_f(r) \quad \text{and} \quad \| \ N(r,\mu_{\psi_2}) \leqslant \frac{\epsilon}{3} T_f(r).$$

Combining with (3.13), we have

$$\| \ T_\psi(r) \leqslant \epsilon T_f(r).$$

Hence, by (3.12) and by the First Main Theorem, for any $\epsilon > 0$ we have

$$N^{[1]}_{f-a_1} \leqslant N_\psi(r) \leqslant \| \, T_\psi(r) + O(1) \leqslant \epsilon T_f(r)$$

(note that $\mathrm{Zero}(f-a_1) = \mathrm{Zero}(g-a_1)$ and $a_j \in \mathcal{R}_f$). Then

$$N(r, D_{(f,g,a_1)}) \leqslant N^{[1]}_{f-a_1}(r) \leqslant \epsilon T_f(r).$$

We get (3.11) in Subcase 1a.

Subcase 1b. $\psi \equiv 0$. Then by (3.12), we have

$$\left(\frac{(f-a_2)'}{f-a_2} - \frac{(f-a_3)'}{f-a_3}\right)(f-a_1) \equiv \left(\frac{(g-a_2)'}{g-a_2} - \frac{(g-a_3)'}{g-a_3}\right)(g-a_1). \tag{3.14}$$

This implies that for any $z_0 \in \mathrm{Zero}(f-a_1) \setminus S_0 = \mathrm{Zero}(g-a_1) \setminus S_0$ we have $\nu_{f-a_1}(z_0) = \nu_{g-a_1}(z_0)$. Therefore,

$$N(r, D_{(f,g,a_1)}) = o\big(T_f(r)\big).$$

We get (3.11) in Subcase 1b.

From (3.7), (3.10), (3.11) we get (i) in Case 1.

Case 2. $\Phi_1 \cdot \Phi_2 \not\equiv 0$.

By (3.7) and Lemma 2.3(ii) (with $\alpha_1 = a_1$, $\alpha_2 = a_3$, $\alpha_3 = a_2$), for any $\epsilon > 0$ we have

$$\| \, T_{\Phi_1 \cdot \Phi_2}(r) \leqslant \epsilon\big(T_f(r) + T_g(r)\big) \tag{3.15}$$

(note that $a_4 = \infty$).

By Lemma 2.2, we have

$$m(r, \Phi_1) = o\big(T_f(r)\big), \qquad m(r, \Phi_2) = o\big(T_f(r)\big).$$

Therefore, by the First Main Theorem, we have

$$\begin{aligned} N^{[1]}_{f-a_1}(r) \leqslant N_{\Phi_2}(r) + o\big(T_f(r)\big) &\leqslant T_{\Phi_2}(r) + o\big(T_f(r)\big) \\ &\leqslant N_{\frac{1}{\Phi_2}}(r) + m(r, \Phi_2) + o\big(T_f(r)\big) \\ &= N_{\frac{1}{\Phi_2}}(r) + o\big(T_f(r)\big). \end{aligned}$$

Combining with Lemma 2.3(i) (with $\Phi = \Phi_2$, $\alpha_1 = a_2$, $\alpha_2 = a_3$, $\alpha_3 = a_1$) we have

$$\begin{aligned} N^{[1]}_{f-a_1}(r) &\leqslant N_{\frac{1}{\Phi_2}}(r) + o\big(T_f(r)\big) \\ &\leqslant N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\mu_f - \mu_g|\big) + o\big(T_f(r)\big) \\ &\leqslant N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) + o\big(T_f(r)\big). \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned} N^{[1]}_{f-a_2}(r) &\leqslant N_{\frac{1}{\Phi_1}}(r) + o\big(T_f(r)\big) \\ &\leqslant N^{[1]}\big(r, |\nu_{f-a_1} - \nu_{g-a_1}|\big) + N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) + o\big(T_f(r)\big). \end{aligned} \tag{3.17}$$

Since $\mathrm{Zero}(f-a_i) = \mathrm{Zero}(g-a_i)$, we have that for any z_0, if $|\nu_{f-a_i}(z_0) - \nu_{g-a_i}(z_0)| \neq 0$ then $\nu_{f-a_i}(z_0) \geqslant 2$ or $\nu_{g-a_i}(z_0) \geqslant 2$. Therefore,

$$N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) \leqslant \sum_{i=3,4} \big({}^{\geqslant 2}N^{[1]}_{f-a_i}(r) + {}^{\geqslant 2}N^{[1]}_{g-a_i}(r)\big).$$

Combining with (3.6), we get

$$\| \, N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r|\nu_{f-a_4} - \nu_{g-a_4}|\big) \leqslant \frac{\epsilon}{3}\big(T_f(r) + T_g(r)\big). \tag{3.18}$$

From (3.16) and (3.18) for any $\epsilon > 0$ we have

$$N^{[1]}_{f-a_1}(r) \leqslant N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big)$$
$$\leqslant N^{[1]}_{f-a_2}(r) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big). \tag{3.19}$$

From (3.17), (3.18) we get

$$N^{[1]}_{f-a_2}(r) \leqslant N^{[1]}\big(r, |\nu_{f-a_1} - \nu_{g-a_1}|\big) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big)$$
$$\leqslant N^{[1]}_{f-a_1}(r) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big). \tag{3.20}$$

By (3.19) and (3.20) for any $\epsilon > 0$, we have

$$N^{[1]}_{f-a_2}(r) \leqslant N^{[1]}_{f-a_1}(r) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big)$$
$$\leqslant N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) + \epsilon\big(T_f(r) + T_g(r)\big) \leqslant N^{[1]}_{f-a_2}(r) + \epsilon\big(T_f(r) + T_g(r)\big)$$

and

$$N^{[1]}_{f-a_1}(r) \leqslant N^{[1]}_{f-a_2}(r) + \frac{\epsilon}{2}\big(T_f(r) + T_g(r)\big)$$
$$\leqslant N^{[1]}\big(r, |\nu_{f-a_1} - \nu_{g-a_1}|\big) + \epsilon\big(T_f(r) + T_g(r)\big) \leqslant N^{[1]}_{f-a_1}(r) + \epsilon\big(T_f(r) + T_g(r)\big).$$

Therefore, for any $\epsilon > 0$ we have

$$\| \; \big|N^{[1]}_{f-a_1}(r) - N^{[1]}_{f-a_2}(r)\big| \leqslant \epsilon\big(T_f(r) + T_g(r)\big), \tag{3.21}$$
$$\| \; N^{[1]}_{f-a_2}(r) - N^{[1]}\big(r, |\nu_{f-a_2} - \nu_{g-a_2}|\big) \leqslant \epsilon\big(T_f(r) + T_g(r)\big) \tag{3.22}$$

and

$$\| \; N^{[1]}_{f-a_1}(r) - N^{[1]}\big(r, |\nu_{f-a_1} - \nu_{g-a_1}|\big) \leqslant \epsilon\big(T_f(r) + T_g(r)\big). \tag{3.23}$$

We now prove the following claim:

Claim. *For any $\epsilon > 0$, we have*

$$\| \; N^{[1]}_{f-a_1}(r) \leqslant \epsilon\big(T_f(r) + T_g(r)\big). \tag{3.24}$$

Proof. Set

$$\phi := \frac{(f-a_3)'}{f-a_3} - \frac{(g-a_3)'}{g-a_3}.$$

If $\phi \equiv 0$, then since $f \not\equiv g$ there exists $c \in \mathbb{C}$, $c \neq 1$ such that $(f - a_3) \equiv c(g - a_3)$. Then $(f - a_i) \equiv c(g - a_i) + (c - 1)(a_i - a_3)$ for $i = 1, 2$. It is easy to see that $\mu_\phi \leqslant 1$ and $\mu_\phi = 0$ on $\{z\colon\ \nu_{f-a_3}(z) = \nu_{g-a_3}(z)\} \cup \{z\colon\ \mu_f(z) = \mu_g(z)\} \setminus \mathrm{Pole}(a_3)$. Therefore,

$$\mu_\phi = \mu^{[1]}_\phi \leqslant \mu_{a_3} + \min\big\{|\nu_{f-a_3} - \nu_{g-a_3}|, 1\big\} + \min\big\{|\mu_f - \mu_g|, 1\big\}$$
$$= \mu_{a_3} + \min\big\{|\nu_{f-a_3} - \nu_{g-a_3}|, 1\big\} + \min\big\{|\nu_{f-a_4} - \mu_{g-a_4}|, 1\big\}.$$

Hence, by (3.6), the First Main Theorem and the Lemma of Logarithmic Derivative, for any $\epsilon > 0$ we have

$$\| \; N_\phi(r) - O(1) \leqslant T_\phi(r) = N(r, \mu_\phi) + m(r, \phi)$$
$$\leqslant N^{[1]}\big(r, |\nu_{f-a_3} - \nu_{g-a_3}|\big) + N^{[1]}\big(r, |\nu_{f-a_4} - \nu_{g-a_4}|\big) + o\big(T_f(r)\big)$$
$$\leqslant \sum_{i=3,4} \big({}^{\geqslant 2}N^{[1]}_{f-a_i}(r) + {}^{\geqslant 2}N^{[1]}_{g-a_i}(r)\big) + o\big(T_f(r)\big) \leqslant \epsilon\big(T_f(r) + T_g(r)\big). \tag{3.25}$$

Suppose that the claim does not hold. Then by (3.21) there exists $E \subset [1, \infty)$ with infinite Lebesgue measure and there exists $\epsilon_0 > 0$ such that

$$N^{[1]}_{f-a_1}(r) > \epsilon_0\big(T_f(r) + T_g(r)\big) \quad \text{and} \quad N^{[1]}_{f-a_2}(r) > \epsilon_0\big(T_f(r) + T_g(r)\big) \tag{3.26}$$

for all $r \in E$.

Let n be a positive integer such that $(n+1)\epsilon_0 > 16$.

By (3.22) and (3.23) for $j = 1, 2$ we have

$$\| \; N^{[1]}_{(a_j;1,1)}(r) \leqslant N^{[1]}_{f-a_j}(r) - N^{[1]}\big(r, |\nu_{f-a_j} - \nu_{g-a_j}|\big) \leqslant \frac{\epsilon_0}{8}\big(T_f(r) + T_g(r)\big). \tag{3.27}$$

By (3.26) and by the First Main Theorem, for $j = 1, 2$ we have

$$\begin{aligned} \|_E \; \epsilon_0\big(T_f(r) + T_g(r)\big) < N^{[1]}_{f-a_j}(r) &= \sum_{i=1}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) + {}^{\geqslant n+1}N^{[1]}_{f-a_j}(r) \\ &\leqslant \sum_{i=1}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) + \frac{1}{n+1}N_{f-a_j}(r) \\ &\leqslant \sum_{i=1}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) + \frac{\epsilon_0}{8}T_f(r), \end{aligned}$$

where ${}^{(i)}N^{[1]}_{f-a_j}(r)$ denotes the counting function (regardless multiplicity) of all points z satisfying $\nu_{f-a_j}(z) = i$ and the notation $\|_E \; P$ means that the assertion P holds for all $r \in E$ excluding a set of finite Lebesgue measure.

Combining with (3.27) and using the First Main Theorem, for $j = 1, 2$ we have

$$\begin{aligned} \Big\|_E \; \frac{7\epsilon_0}{8}\big(T_f(r) + T_g(r)\big) &\leqslant \sum_{i=1}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) \\ &\leqslant N^{[1]}_{(a_j;1,1)}(r) + \sum_{i\geqslant 2} {}^{(i)}N^{[1]}_{g-a_j}(r) + \sum_{i=2}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) \\ &\leqslant N^{[1]}_{(a_j;1,1)}(r) + \sum_{i=2}^{n} {}^{(i)}N^{[1]}_{g-a_j}(r) + \frac{1}{n+1}N_{g-a_j}(r) + \sum_{i=2}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r) \\ &\leqslant \frac{\epsilon_0}{8}\big(T_f(r) + T_g(r)\big) + \sum_{i=2}^{n} {}^{(i)}N_{g-a_j}(r) + \frac{\epsilon_0}{8}T_g(r) + \sum_{i=2}^{n} {}^{(i)}N^{[1]}_{f-a_j}(r). \end{aligned}$$

This implies that for each $j \in \{1, 2\}$, there exist $n_j \in \{2, \dots, n\}$ such that

$$\|_E \; {}^{(n_j)}N^{[1]}_{f-a_j}(r) \geqslant \frac{\epsilon_0}{16n}\big(T_f(r) + T_g(r)\big) \quad \text{or} \quad \|_E \; {}^{(n_j)}N^{[1]}_{g-a_j}(r) \geqslant \frac{\epsilon_0}{16n}\big(T_f(r) + T_g(r)\big). \tag{3.28}$$

Combining with (3.5), for each $j \in \{1, 2\}$, there exist $n_j \geqslant 2$ and $\epsilon_1 > 0$, such that

$$\|_E \; N^{[1]}_{(a_j;n_j,1)}(r) \geqslant \epsilon_1\big(T_f(r) + T_g(r)\big) \quad \text{or} \quad \|_E \; N^{[1]}_{(a_j;1,n_j)}(r) \geqslant \epsilon_1\big(T_f(r) + T_g(r)\big). \tag{3.29}$$

Noting that f and g are symmetric, for the proof of the claim, we only need to consider two following cases:

Subcase 2a.

$$\|_E \; N^{[1]}_{(a_1;n_1,1)}(r) \geqslant \epsilon_1\big(T_f(r) + T_g(r)\big) \quad \text{and} \quad \|_E \; N^{[1]}_{(a_2;n_2,1)}(r) \geqslant \epsilon_1\big(T_f(r) + T_g(r)\big). \tag{3.30}$$

For any $z_0 \in \{z\colon\ \nu_{f-a_1}(z_0) = n_1,\ \nu_{g-a_1}(z) = 1\} \setminus S_0$, we write $f - a_1 = (z - z_0)^{n_1}u(z)$ and $g - a_1 = (z - z_0)v(z)$ (on a neighborhood of z_0) where u, v are holomorphic functions and $u(z_0), v(z_0) \neq 0$. Then, by an easy computation, we have

$$\begin{aligned} &\phi(z_0) = \frac{-v}{a_1 - a_3}(z_0), \qquad \big((z - z_0)\cdot \Phi_1\big)(z_0) = (1 - n_1)\big(a_1(z_0) - a_2(z_0)\big), \\ &\frac{\Phi_2}{z - z_0}(z_0) = -v(z_0)\left(\frac{v(a_3 - a_2) + a_1'(a_3 - a_2)}{(a_1 - a_2)(a_1 - a_3)}(z_0) + \frac{a_2'}{a_1 - a_2}(z_0) - \frac{a_3'}{a_1 - a_3}(z_0)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} (\Phi_1 \cdot \Phi_2)(z_0) &= (1 - n_1)\phi\big(\phi(a_3 - a_1)(a_3 - a_2) + a_1'(a_3 - a_2) + a_2'(a_1 - a_3) + a_3'(a_2 - a_1)\big)(z_0) \\ &= (1 - n_1)\phi\big(\phi(a_3 - a_1)(a_3 - a_2) + a_3'(a_2 - a_1)\big)(z_0) \end{aligned} \tag{3.31}$$

(note that a_1, a_2 are constants).

This implies that

$$(\Phi_1 \cdot \Phi_2) \equiv (1-n_1)\phi\big(\phi(a_3-a_1)(a_3-a_2)+a_3'(a_2-a_1)\big). \tag{3.32}$$

Suppose that (3.32) does not hold. Then by (3.15), (3.25), (3.30), (3.31) and by the First Main Theorem, for any $\epsilon > 0$ we have

$$\begin{aligned} \|_E\ \epsilon\big(T_f(r)+T_g(r)\big) &\geqslant T_{\Phi_1\cdot\Phi_2}(r)+2T_\phi(r)+o\big(T_f(r)\big) \\ &\geqslant T_{\Phi_1\cdot\Phi_2-(1-n_1)\phi(\phi(a_3-a_1)(a_3-a_2)+a_3'(a_2-a_1))}(r)+o\big(T_f(r)\big) \\ &\geqslant N_{\Phi_1\cdot\Phi_2-(1-n_1)\phi(\phi(a_3-a_1)(a_3-a_2)+a_3'(a_2-a_1))}(r) \\ &\overset{(3.31)}{\geqslant} N^{[1]}_{(a_1;n_1,1)}(r) \geqslant \epsilon_1\big(T_f(r)+T_g(r)\big). \end{aligned}$$

This is a contradiction. Thus, we get (3.32).

For any $z_1 \in \{z\colon\ \nu_{f-a_2}(z)=n_2,\ \nu_{g-a_2}=1\} \setminus S_0$, similarly to (3.31), we have

$$\begin{aligned} (\Phi_1\cdot\Phi_2)(z_1) &= (1-n_2)\phi\big(\phi(a_3-a_2)(a_3-a_1)+a_2'(a_3-a_1)+a_1'(a_2-a_3)+a_3'(a_1-a_2)\big)(z_1) \\ &= (1-n_2)\phi\big(\phi(a_3-a_2)(a_3-a_1)+a_3'(a_1-a_2)\big)(z_1). \end{aligned}$$

Therefore, similarly to (3.32), we have

$$(\Phi_1 \cdot \Phi_2) \equiv (1-n_2)\phi\big(\phi(a_3-a_2)(a_3-a_1)+a_3'(a_1-a_2)\big). \tag{3.33}$$

Noting that $\phi \not\equiv 0$, by (3.32) and (3.33) we have

$$(n_1-n_2)\phi = (2-n_1-n_2)\frac{a_3'(a_2-a_1)}{(a_3-a_2)(a_3-a_1)}. \tag{3.34}$$

On the other hand a_3 is not constant and $n_1, n_2 \geqslant 2$. Hence, by (3.34) we have $n_1 \neq n_2$ and

$$(n_1-n_2)\ln'\left(\frac{f-a_3}{g-a_3}\right) \equiv (2-n_1-n_2)\ln'\left(\frac{a_3-a_2}{a_3-a_1}\right)$$

(note that a_1, a_2 are constants).

Then $n_1 \neq n_2$ and there exists a constant c such that

$$\left(\frac{f-a_3}{g-a_3}\right)^{n_1-n_2} \equiv c\cdot\left(\frac{a_3-a_2}{a_3-a_1}\right)^{2-n_1-n_2}.$$

Therefore, since $a_j \in \mathcal{R}_f$, $n_1 \neq n_2$, $2-n_1-n_2 \neq 0$ and by (3.30), we have

$$\begin{aligned} \|_E\ o\big(T_f(r)\big) &= N^{[1]}_{c\cdot(\frac{a_3-a_2}{a_3-a_1})^{2-n_1-n_2}-1}(r) \\ &= N^{[1]}_{(\frac{f-a_3}{g-a_3})^{n_1-n_2}-1}(r) \geqslant N^{[1]}_{(a_1;n_1,1)}(r) - o\big(T_f(r)\big) \\ &\geqslant \epsilon_1\big(T_f(r)+T_g(r)\big) - o\big(T_f(r)\big). \end{aligned}$$

This is a contradiction.

Subcase 2b.

$$\|_E\ N^{[1]}_{(a_1;n_1,1)}(r) \geqslant \epsilon_1\big(T_f(r)+T_g(r)\big) \quad \text{and} \quad \|_E\ N^{[1]}_{(a_2;1,n_2)}(r) \geqslant \epsilon_1\big(T_f(r)+T_g(r)\big). \tag{3.35}$$

By an argument similar to Subcase 2a, we have

$$(\Phi_1 \cdot \Phi_2) \equiv (1-n_1)\phi\big(\phi(a_3-a_1)(a_3-a_2)+a_3'(a_2-a_1)\big), \tag{3.36}$$

and

$$(\Phi_1 \cdot \Phi_2) \equiv (1-n_2)\phi\big(\phi(a_3-a_1)(a_3-a_2)+a_3'(a_2-a_1)\big). \tag{3.37}$$

If $n_1 \neq n_2$, then by (3.36) and (3.37) and since $\phi \not\equiv 0$, we get

$$\phi \equiv \frac{a_3'(a_1-a_2)}{(a_3-a_1)(a_3-a_2)}.$$

Then, since $a_1, a_2 \in \mathbb{C}$ we have

$$\ln'\left(\frac{f-a_3}{g-a_3}\right) \equiv \ln'\left(\frac{a_3-a_1}{a_3-a_2}\right).$$

Hence, there exists a constant c_1 such that

$$\frac{f-a_3}{g-a_3} \equiv c_1 \cdot \frac{a_3-a_1}{a_1-a_2}.$$

Therefore, since $a_j \in \mathcal{R}_f$ and by (3.35), we have

$$\begin{aligned}\|_E\, o\big(T_f(r)\big) &= N^{[1]}_{c_1\cdot\frac{a_3-a_1}{a_3-a_2}-1}(r)\\ &= N^{[1]}_{\frac{f-a_3}{g-a_3}-1}(r) \geqslant N^{[1]}_{(a_1;n_1,1)}(r) - o\big(T_f(r)\big)\\ &\geqslant \epsilon_1\big(T_f(r)+T_g(r)\big) - o\big(T_f(r)\big).\end{aligned}$$

This is a contradiction.

We now assume that $n_1 = n_2$. For any $z_0 \in \{z\colon\ \nu_{f-a_1}(z) = n_1,\ \nu_{g-a_1}(z) = 1\} \setminus S_0$, on a neighborhood of z_0 we write $f - a_1 = (z-z_0)^{n_1}u(z)$, $g - a_1 = (z-z_0)v(z)$ where u, v are holomorphic functions and $u(z_0), v(z_0) \neq 0$.

By an easy computation we have

$$\phi(z_0) = \frac{-v}{a_1-a_3}(z_0), \qquad \big((g-a_1)\cdot\Phi_1\big)(z_0) = (1-n_1)\big(a_1(z_0)-a_2(z_0)\big)v(z_0).$$

Then

$$\big((g-a_1)\Phi_1\big)(z_0) = (1-n_1)\big((f-a_2)\phi(a_3-a_1)\big)(z_0). \tag{3.38}$$

Similarly, for any $z_1 \in \{z\colon\ \nu_{f-a_2}(z) = 1,\ \nu_{g-a_2}(z) = n_1 = n_2\} \setminus S_0$, we have

$$\big((f-a_2)\Phi_2\big)(z_1) = (1-n_1)\big((g-a_1)\phi(a_3-a_2)\big)(z_1). \tag{3.39}$$

Set

$$h_1 := \frac{(g-a_1)\Phi_1}{(1-n_1)(f-a_2)\phi(a_3-a_1)}, \qquad h_2 := \frac{(f-a_2)\Phi_2}{(1-n_1)(g-a_1)\phi(a_3-a_2)}, \qquad h_3 := \frac{f-a_3}{g-a_3}.$$

It is easy to see that

$$T_{h_i}(r) \leqslant O\big(T_f(r)+T_g(r)\big) \quad \text{for all } i = 1,2,3. \tag{3.40}$$

By (3.35), (3.38), (3.39) and the First Main Theorem, for $i \in \{1,2\}$ and $j \in \{2,3\}$ we have

$$\begin{aligned}T_{h_i}(r) &\geqslant N^{[1]}_{(h_1=1=h_3)}(r) - O(1) \geqslant N^{[1]}_{(a_1;n_1,1)}(r) - o\big(T_f(r)\big) \geqslant \frac{\epsilon_1}{2}\big(T_f(r)+T_g(r)\big),\\ T_{h_j}(r) &\geqslant N^{[1]}_{(h_2=1=h_3)}(r) - O(1) \geqslant N^{[1]}_{(a_2;1,n_2)}(r) - o\big(T_f(r)\big) \geqslant \frac{\epsilon_1}{2}\big(T_f(r)+T_g(r)\big),\end{aligned} \tag{3.41}$$

where $N^{[1]}_{(h_i=1=h_j)}(r)$ denotes the counting function of those common 1-points regardless multiplicity of h_i, h_j.

Combining with (3.40) we have $T_{h_i}(r) = O(T_f(r)+T_g(r))$ for $i \in \{1,2,3\}$ and there exists $\epsilon_2 > 0$ such that

$$N^{[1]}_{(h_1=1=h_3)}(r) \geqslant \epsilon_2\big(T_{h_1}(r)+T_{h_2}(r)\big) \quad\text{and}\quad N^{[1]}_{(h_2=1=h_3)}(r) \geqslant \epsilon_2\big(T_{h_2}(r)+T_{h_3}(r)\big).$$

On the other hand by (3.18), (3.25), and Lemma 2.3(iii), (iv), for any $\epsilon > 0$ we have

$$\|\ N^{[1]}_{h_i}(r) + N^{[1]}_{\frac{1}{h_i}}(r) \leqslant \epsilon\big(T_f(r)+T_g(r)\big) \quad (i = 1,2).$$

By (3.7) we also have

$$N^{[1]}_{h_3}(r) + N^{[1]}_{\frac{1}{h_3}}(r) \leqslant N(r, D_{(f,g,a_3)}) \leqslant \epsilon\big(T_f(r)+T_g(r)\big).$$

Therefore, by Lemma 2.1, there exist two pairs of integers $(p_1,q_1), (p_2,q_2) \neq (0,0)$ such that $h_1^{p_1} \equiv h_3^{q_1}$ and $h_2^{p_2} \equiv h_3^{q_2}$. This implies that

$$\left(\frac{\Phi_1\Phi_2}{(n_1-1)^2(a_3-a_1)(a_3-a_2)\phi^2}\right)^{p_1p_2} \equiv (h_1h_2)^{p_1p_2} \equiv h_3^{p_1q_2+p_2q_1}. \tag{3.42}$$

By (3.15) and (3.25), for any $\epsilon > 0$ we have

$$T_{\frac{\Phi_1\Phi_2}{(n_1-1)^2(a_3-a_1)(a_3-a_2)\phi^2}}(r) \leqslant \epsilon\big(T_f(r)+T_g(r)\big).$$

On the other hand by (3.41) we have

$$\| \; T_{h_3}(r) \geqslant \frac{\epsilon_1}{2}\big(T_f(r)+T_g(r)\big). \tag{3.43}$$

Therefore, from (3.42), we have $p_1q_2 + p_2q_1 = 0$. Then there exists a constant c such that

$$\Phi_1\Phi_2 \equiv c(n_1-1)^2(a_3-a_1)(a_3-a_2)\phi^2.$$

Combining with (3.36) we have

$$\big(c(1-n_1)-1\big)\phi \equiv \frac{a_3'(a_2-a_1)}{(a_3-a_1)(a_3-a_2)}.$$

This implies that

$$\big(c(1-n_1)-1\big)\ln'\left(\frac{f-a_3}{g-a_3}\right) \equiv \ln'\left(\frac{a_3-a_2}{a_3-a_1}\right)$$

(note that a_1, a_2 are constants).

Then there exists a constant $K \neq 0$ such that

$$h_3^{c(1-n_1)-1} \equiv \left(\frac{f-a_3}{g-a_3}\right)^{c(1-n_1)-1} \equiv K\frac{a_3-a_2}{a_3-a_1}. \tag{3.44}$$

On the other hand, since $a_1 \neq a_2 \in \mathbb{C}$ and $a_3 \notin \mathbb{C}$, we get that $\frac{a_3-a_2}{a_3-a_1}$ is not constant. Therefore, by (3.43) and (3.44) we have $c(1-n_1)-1 \neq 0$ and

$$\left\| \; \frac{|c(1-n_1)-1|\epsilon_1}{2}\big(T_f(r)+T_g(r)\big) \leqslant T_{h_3^{c(1-n_1)-1}}(r) = T_{K\frac{a_3-a_1}{a_3-a_2}}(r) = o\big(T_f(r)\big).\right.$$

This is a contradiction.

The proof of the claim has been completed. □

Note that a_1, a_2 are symmetric, so similarly to (3.24) for any $\epsilon > 0$ we have

$$\| \; N^{[1]}_{f-a_2}(r) \leqslant \epsilon\big(T_f(r)+T_g(r)\big). \tag{3.45}$$

By (3.24) and (3.45) for $j = 1, 2$ we have

$$\| \; N(r, D_{(f,g,a_j)}) \leqslant N^{[1]}_{f-a_j}(r) \leqslant \epsilon\big(T_f(r)+T_g(r)\big). \tag{3.46}$$

By (3.7) and (3.46), assertion (i) holds in Case 2.

We now prove the assertions (ii), (iii).

For two meromorphic functions h, u on $\mathbb{C}$ with reduced representations $h = (h_0 : h_1)$, $u = (u_0 : u_1)$, we set $(h, u) = h_0u_0 + h_1u_1$.

For each $j \in \{1, \ldots, 4\}$, we set $\tilde{a}_{j0} := a_{j1}$, $\tilde{a}_{j1} := -a_{j0}$ and $\tilde{a}_j := (\tilde{a}_{j0} : \tilde{a}_{j1}) \in \mathcal{R}_f$.

It is easy to see that for $1 \leqslant j \leqslant 4$,

$$\nu_{f-a_j} = \nu_{(f,\tilde{a}_j)}, \qquad \nu_{g-a_j} = \nu_{(g,\tilde{a}_j)}.$$

Therefore, for all $1 \leqslant j \leqslant 4$,

$$\nu_{(f,\tilde{a}_j)} = \nu_{(g,\tilde{a}_j)} \quad \text{on } \big\{z\colon D_{(f,g,a_j)}(z) = 0\big\}. \tag{3.47}$$

Define functions $h_j := \frac{(f,\tilde{a}_j)}{(g,\tilde{a}_j)}$, $j \in \{1, \ldots, 4\}$. We have

$$\tilde{a}_{j0}f_0 + \tilde{a}_{j1}f_1 - h_j\tilde{a}_{j0}g_0 - h_j\tilde{a}_{j1}g_1 = 0, \quad j \in \{1, \ldots, 4\}.$$

Therefore, $\det(\tilde{a}_{j0}, \tilde{a}_{j1}, -h_j\tilde{a}_{j0}, -h_j\tilde{a}_{j1},\ 1 \leqslant j \leqslant 4) \equiv 0$.

For each $j \in \{1, \ldots, 4\}$, we fix an index $k_j \in \{0, 1\}$ such that $\tilde{a}_{jk_j} \not\equiv 0$. For each pair $I = \{u, v\}$ with $1 \leqslant u < v \leqslant 4$, we define $h_I = h_u h_v$ and

$$A_I = (-1)^{1+u+v} \cdot \frac{\begin{vmatrix} \tilde{a}_{u0} & \tilde{a}_{u1} \\ \tilde{a}_{v0} & \tilde{a}_{v1} \end{vmatrix} \cdot \begin{vmatrix} \tilde{a}_{u'0} & \tilde{a}_{u'1} \\ \tilde{a}_{v'0} & \tilde{a}_{v'1} \end{vmatrix}}{\tilde{a}_{1k_1}\tilde{a}_{2k_2}\tilde{a}_{3k_3}\tilde{a}_{4k_4}}$$

where $\{u', v'\} = \{1, \ldots, 4\} \setminus \{u, v\}$ and $u' < v'$. We have $A_I \in \mathcal{R}_f$ and $A_I \not\equiv 0$. Set $\mathcal{L} = \{I \subset \{1, \ldots, 4\}\colon \#I = 2\}$, then $\#\mathcal{L} = 6$. By the Laplace Expansion Theorem, we have

$$\sum_{I \in \mathcal{L}} A_I h_I \equiv 0. \tag{3.48}$$

We introduce an equivalence relation on $\mathcal{L}$ as follows: $I \backsim J$ if and only if $\frac{h_I}{h_J} \in \mathcal{R}_f$.

Set $\{L_1, \ldots, L_s\} = \mathcal{L} \diagup \backsim$ $(s \leqslant 6)$. For each $v \in \{1, \ldots, s\}$, choose $I_v \in L_v$ and set

$$\sum_{I \in L_v} A_I h_I = B_v h_{I_v}, \quad B_v \in \mathcal{R}_f.$$

Then (3.48) can be written as

$$\sum_{v=1}^{s} B_v h_{I_v} \equiv 0. \tag{3.49}$$

Case A. There exists some $B_v \not\equiv 0$. We may assume that $B_v \not\equiv 0$, for all $v \in \{1, \ldots, l\}$, $B_v \equiv 0$ for all $v \in \{l+1, \ldots, s\}$ $(1 \leqslant l \leqslant s)$. By (3.49) we have

$$\sum_{v=1}^{l} B_v h_{I_v} \equiv 0. \tag{3.50}$$

Denote by $\mathcal{P}$ the set of all positive integers $p \leqslant l$ such that there exist a subset $K_p \subseteq \{1, \ldots, l\}$, $\#K_p = p$ and nonzero constants $\{c_i\}_{i \in K_p}$ with $\sum_{i \in K_p} c_i B_i h_{I_i} \equiv 0$. By (3.50), we have $l \in \mathcal{P}$. Let t be the smallest integer in $\mathcal{P}$ $(t \leqslant l \leqslant 6)$. We may assume that $K_t = \{1, \ldots, t\}$. Then there exist nonzero constants c_v $(v = 1, \ldots, t)$ such that

$$\sum_{v=1}^{t} c_v B_v h_{I_v} \equiv 0. \tag{3.51}$$

Since $\frac{h_{I_i}}{h_{I_j}} \notin \mathcal{R}_f$ and $h_{I_i} \not\equiv 0$ for all $1 \leqslant i \neq j \leqslant t$, we have $t \geqslant 3$.

Without loss of generality, we may assume that

$$T_{\frac{h_{I_1}}{h_{I_2}}}(r) = \max\left\{T_{\frac{h_{I_1}}{h_{I_2}}}(r), T_{\frac{h_{I_2}}{h_{I_3}}}(r), T_{\frac{h_{I_3}}{h_{I_1}}}(r)\right\} \tag{3.52}$$

for all $r \in \mathcal{A}$, where $\mathcal{A}$ is a subset of $[1, +\infty)$ with infinite Lebesgue measure.

It is easy to see that

$$\big((I_1 \cup I_2) \setminus (I_1 \cap I_2)\big) \cap \big((I_2 \cup I_3) \setminus (I_2 \cap I_3)\big) \cap \big((I_3 \cup I_1) \setminus (I_3 \cap I_1)\big) = \emptyset.$$

On the other hand

$$\frac{h_{I_i}}{h_{I_j}} = 1 \quad \text{on} \bigcup_{k=1}^{4} \operatorname{Zero}(f, \tilde{a}_k) \setminus \bigcup_{k \in (I_i \cup I_j) \setminus (I_i \cap I_j)} \operatorname{Zero}(f, \tilde{a}_k) \quad \text{for all } 1 \leqslant i \neq j \leqslant 3.$$

Hence, by the First and the Second Main Theorems, we have

$$\begin{aligned} \Big\| \ T_{\frac{h_{I_1}}{h_{I_2}}}(r) + T_{\frac{h_{I_2}}{h_{I_3}}}(r) + T_{\frac{h_{I_3}}{h_{I_1}}}(r) &\geqslant N_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N_{\frac{h_{I_2}}{h_{I_3}}-1}(r) + N_{\frac{h_{I_3}}{h_{I_1}}-1}(r) - O(1) \\ &\geqslant \sum_{i=1}^{4} N^{[1]}_{(f,\tilde{a}_i)}(r) - o\big(T_f(r)\big) \geqslant (2-\epsilon) T_f(r) \end{aligned} \tag{3.53}$$

(note that $\{z\colon (f, \tilde{a}_i)(z) = (f, \tilde{a}_j)(z) = 0\} \subset \{z\colon \tilde{a}_i(z) = \tilde{a}_j(z)\}$ and $\tilde{a}_i - \tilde{a}_j \in \mathcal{R}_f$ for all $1 \leqslant i < j \leqslant 4$).

Similarly,

$$\| \, T_{\frac{h_{I_1}}{h_{I_2}}}(r) + T_{\frac{h_{I_2}}{h_{I_3}}}(r) + T_{\frac{h_{I_3}}{h_{I_1}}}(r) \geqslant (2-\epsilon) T_g(r). \tag{3.54}$$

Define the meromorphic mapping $H := (c_1 B_1 h_{I_1} : \cdots : c_{t-1} B_{t-1} h_{I_{t-1}}) : \mathbb{C} \to \mathbb{C}P^{(t-2)}$. Since $t = \min \mathcal{P}$, H is linearly nondegenerate.

By (3.52), (3.53) and (3.54) we have

$$T_H(r) + o\big(T_f(r)\big) \geqslant T_{\frac{h_{I_1}}{h_{I_2}}}(r) \geqslant \frac{1}{3}\big(T_{\frac{h_{I_1}}{h_{I_2}}}(r) + T_{\frac{h_{I_2}}{h_{I_3}}}(r) + T_{\frac{h_{I_3}}{h_{I_1}}}(r)\big) \geqslant \frac{2-\epsilon}{6}\big(T_f(r) + T_g(r)\big) \tag{3.55}$$

for all $r \in \mathcal{A}_1$, where $\mathcal{A}_1$ is a subset of $[1, +\infty)$ with infinite Lebesgue measure.

Let $(\frac{c_1 B_1 h_{I_1}}{u} : \cdots : \frac{c_{t-1} B_{t-1} h_{I_{t-1}}}{u})$ be a reduced representation of H, where u is a meromorphic function on $\mathbb{C}$. It is clear that a zero of $\frac{c_i B_i h_{I_i}}{u}$ $(i = 1, \ldots, t)$ is a zero or a pole of some $c_j B_j h_{I_j}$, $j \in \{1, \ldots, t\}$. Hence, for each $i \in \{1, \ldots, t\}$, we have

$$\begin{aligned} N^{[1]}_{\frac{c_i B_i h_{I_i}}{u}}(r) &\leqslant \sum_{j=1}^{t} N^{[1]}_{c_j B_j h_{I_j}}(r) + \sum_{j=1}^{t} N^{[1]}_{\frac{1}{c_j B_j h_{I_j}}}(r) \\ &= \sum_{j=1}^{t} N^{[1]}_{h_{I_j}}(r) + \sum_{j=1}^{t} N^{[1]}_{\frac{1}{h_{I_j}}}(r) + o\big(T_f(r)\big). \end{aligned}$$

On the other hand, it is clear that $\nu_{h_{I_j}} = \nu_{\frac{1}{h_{I_j}}} = 0$ on $\bigcup_{i=1}^{4}\{z\colon\ \nu_{(f,\tilde{a}_i)}(z) = 1 = \nu_{(g,\tilde{a}_i)}(z)\} \overset{(3.47)}{=} \{z\colon\ D_{(f,g,a_j)}(z) = 0\}$. Hence, we have

$$\begin{aligned} \sum_{i=1}^{t} N^{[1]}_{\frac{c_i B_i h_{I_i}}{u}}(r) &\leqslant t \sum_{j=1}^{t} N^{[1]}_{h_{I_j}}(r) + t \sum_{j=1}^{t} N^{[1]}_{\frac{1}{h_{I_j}}}(r) + o\big(T_f(r)\big) \\ &\leqslant t^2 \sum_{i=1}^{4} N^{[1]}(r, D_{(f,g,a_j)}) + o\big(T_f(r)\big). \end{aligned} \tag{3.56}$$

By assertion (i) and (3.56), for any $\epsilon > 0$ we have

$$\begin{aligned} \sum_{i=1}^{t} N^{[t-2]}_{\frac{c_i B_i h_{I_i}}{u}}(r) &\leqslant (t-2) \sum_{i=1}^{t} N^{[1]}_{\frac{c_i B_i h_{I_i}}{u}}(r) \\ &\leqslant t^2 (t-2)\epsilon\big(T_f(r) + T_g(r)\big) + o\big(T_f(r)\big). \end{aligned}$$

Hence, by (3.51) and the Second Main Theorem, we have

$$\begin{aligned} \| \, T_H(r) &\leqslant \sum_{i=1}^{t-1} N^{[t-2]}_{\frac{c_i B_i h_{I_i}}{u}}(r) + N^{[t-2]}_{\sum_{i=1}^{t-1} \frac{c_i B_i h_{I_i}}{u}}(r) + o\big(T_H(r)\big) = \sum_{i=1}^{t} N^{[t-2]}_{\frac{c_i B_i h_{I_i}}{u}}(r) + o\big(T_H(r)\big) \\ &\leqslant t^2 (t-2)\epsilon\big(T_f(r) + T_g(r)\big) + o\big(T_f(r)\big) + o\big(T_H(r)\big). \end{aligned}$$

Combining with (3.55), we get that for each $\epsilon > 0$, there exists a subset $\mathcal{A}_2$ of $[1, +\infty)$ with infinite Lebesgue measure such that

$$\frac{2-\epsilon}{6}\big(T_f(r) + T_g(r)\big) \leqslant 2t^2 (t-2)\epsilon\big(T_f(r) + T_g(r)\big)$$

for all $r \in \mathcal{A}_2$. This is a contradiction.

Case B. $B_v \equiv 0$ for all $v \in \{1, \ldots, s\}$. Then, since $A_I h_I \not\equiv 0$ $(I \in \mathcal{L})$ we have $\# L_v \geqslant 2$ for all $v \in \{1, \ldots, s\}$. This implies that for each $I = \{u, v\} \subset \{1, \ldots, 4\}$ there exists some $J := \{u', v'\} \subset \{1, \ldots, 4\}$, $J \neq I$ such that

$$\frac{h_I}{h_J} \in \mathcal{R}^*_f. \tag{3.57}$$

Let $\mathcal{M}^*$ be the abelian multiplication group of all nonzero meromorphic functions on $\mathbb{C}$. We have that $\mathcal{R}^*_f = \mathcal{M}^* \cap \mathcal{R}_f$ is a subgroup of $\mathcal{M}^*$ and the multiplication group $\mathcal{G} := \mathcal{M}^* \big/ \mathcal{R}^*_f$ is a torsion free abelian group. We denote by $[h]$ the class in $\mathcal{G}$ containing $h \in \mathcal{M}^*$. Then by (3.57) we have that $A := (h_1, \dots, h_4)$ has the property $P_{4,2}$. By Theorem 2.1, there exists $\{u, v\} \subset \{1, \dots, 4\}$ such that $\frac{h_u}{h_v} \in \mathcal{R}_f^*$. Without loss of generality, we may assume that $\{u, v\} = \{1, 2\}$. Then there exists $\alpha \in \mathcal{R}_f^*$, $\alpha \not\equiv 1$ such that

$$\frac{(f, \tilde{a}_1)}{(f, \tilde{a}_2)} = \alpha \frac{(g, \tilde{a}_1)}{(g, \tilde{a}_2)}. \tag{3.58}$$

Set $D_j := \mathrm{Zero}(f, \tilde{a}_j) = \mathrm{Zero}(g, \tilde{a}_j)$. Then $f = a_j = g$ on D_j for all $1 \leqslant j \leqslant 4$. Therefore, $\alpha = 1$ on $(D_3 \cup D_4) \setminus (D_1 \cup D_2)$. On the other hand $D_i \cap D_j \subset \{z\colon\ \tilde{a}_i(z) = \tilde{a}_j(z)\}$ and $\tilde{a}_i - \tilde{a}_j \in \mathcal{R}_f$ for all $1 \leqslant i < j \leqslant 4$. Hence, by the First Main Theorem, we have

$$N^{[1]}_{(f,\tilde{a}_3)}(r) + N^{[1]}_{(f,\tilde{a}_4)}(r) \leqslant N_{\alpha-1}(r) + o\big(T_f(r)\big) \leqslant T_\alpha(r) + o\big(T_f(r)\big) = o\big(T_f(r)\big).$$

This implies that $\tilde{a}_3, \tilde{a}_4 \in \mathcal{A}_f$. On the other hand we have $\#\mathcal{A}_f \leqslant 2$. Hence, we get $\{\tilde{a}_1, \dots, \tilde{a}_4\} \cap \mathcal{A}_f = \{\tilde{a}_3, \tilde{a}_4\}$. This implies that $\{a_1, \dots, a_4\} \cap \mathcal{A}_f = \{a_3, a_4\}$.

We now prove

$$\frac{(a_4, \tilde{a}_1)}{(a_4, \tilde{a}_2)} = \alpha \frac{(a_3, \tilde{a}_1)}{(a_3, \tilde{a}_2)} \quad \text{and} \quad \frac{(a_3, \tilde{a}_1)}{(a_3, \tilde{a}_2)} = \alpha \frac{(a_4, \tilde{a}_1)}{(a_4, \tilde{a}_2)}. \tag{3.59}$$

Set $F := \frac{(f,\tilde{a}_1)\tilde{a}_{2k_2}}{(f,\tilde{a}_2)\tilde{a}_{1k_1}}$, $G := \frac{(g,\tilde{a}_1)\tilde{a}_{2k_2}}{(g,\tilde{a}_2)\tilde{a}_{1k_1}}$, $b_1 := \frac{(a_3,\tilde{a}_1)\tilde{a}_{2k_2}}{(a_3,\tilde{a}_2)\tilde{a}_{1k_1}}$, $b_2 := \frac{(a_4,\tilde{a}_1)\tilde{a}_{2k_2}}{(a_4,\tilde{a}_2)\tilde{a}_{1k_1}}$, $b_3 := \alpha \frac{(a_3,\tilde{a}_1)\tilde{a}_{2k_2}}{(a_3,\tilde{a}_2)\tilde{a}_{1k_1}}$, $b_4 := \alpha \frac{(a_4,\tilde{a}_1)\tilde{a}_{2k_2}}{(a_4,\tilde{a}_2)\tilde{a}_{1k_1}}$. Then by (3.58), we have $F = \alpha G$. Since $\alpha, \{\tilde{a}_i\}_{i=1}^4 \in \mathcal{R}_f$, it is easy to see that $T_F(r) = T_f(r) + o(T_f(r))$, $T_G(r) = T_g(r) + o(T_g(r))$ and $b_1, b_2, b_3, b_4 \in \mathcal{R}_f$.

By an easy computation, we get that

$$\nu_{F-b_1} = \nu_{\frac{(f,\tilde{a}_3)(a_1,\tilde{a}_2)}{(f,\tilde{a}_2)(a_3,\tilde{a}_2)} \cdot \frac{\tilde{a}_{2k_2}}{\tilde{a}_{1k_1}}}, \qquad \nu_{F-b_2} = \nu_{\frac{(f,\tilde{a}_4)(a_1,\tilde{a}_2)}{(f,\tilde{a}_2)(a_4,\tilde{a}_2)} \cdot \frac{\tilde{a}_{2k_2}}{\tilde{a}_{1k_1}}},$$

$$\nu_{F-b_3} = \nu_{\alpha(G-b_1)} = \nu_{\alpha \frac{(g,\tilde{a}_3)(a_1,\tilde{a}_2)}{(g,\tilde{a}_2)(a_3,\tilde{a}_2)} \cdot \frac{\tilde{a}_{2k_2}}{\tilde{a}_{1k_1}}}, \quad \text{and} \quad \nu_{F-b_4} = \nu_{\alpha(G-b_2)} = \nu_{\alpha \frac{(g,\tilde{a}_4)(a_1,\tilde{a}_2)}{(g,\tilde{a}_2)(a_4,\tilde{a}_2)} \cdot \frac{\tilde{a}_{2k_2}}{\tilde{a}_{1k_1}}}.$$

$*$ If $b_2 \not\equiv b_3$, then b_1, b_2, b_3 are distinct. By the Second Main Theorem, for each $\epsilon > 0$, we have

$$\begin{aligned} \big\| \ (1-\epsilon)T_f(r) + o\big(T_f(r)\big) = (1-\epsilon)T_F(r) &\leqslant \sum_{i=1}^{3} N^{[1]}_{F-b_i}(r) \\ &\leqslant N^{[1]}_{(f,\tilde{a}_3)}(r) + N^{[1]}_{(f,\tilde{a}_4)}(r) + N^{[1]}_{(g,\tilde{a}_3)}(r) + o\big(T_f(r)\big) = o\big(T_f(r)\big) \end{aligned}$$

(note that $\tilde{a}_3, \tilde{a}_4 \in \mathcal{A}_f$). This is a contradiction. Hence, $b_2 \equiv b_3$.

$*$ If $b_1 \not\equiv b_4$, then b_1, b_2, b_4 are distinct. By the Second Main Theorem, for each $\epsilon > 0$, we have

$$\begin{aligned} \big\| \ (1-\epsilon)T_f(r) + o\big(T_f(r)\big) = (1-\epsilon)T_F(r) &\leqslant N^{[1]}_{F-b_1}(r) + N^{[1]}_{F-b_2}(r) + N^{[1]}_{F-b_4}(r) \\ &\leqslant N^{[1]}_{(f,\tilde{a}_3)}(r) + N^{[1]}_{(f,\tilde{a}_4)}(r) + N^{[1]}_{(g,\tilde{a}_4)}(r) + o\big(T_f(r)\big) = o\big(T_f(r)\big). \end{aligned}$$

This is a contradiction. Hence, $b_1 \equiv b_4$.

Since $b_1 \equiv b_4$, $b_2 = b_3$, we get (3.59). Since $\alpha \not\equiv 1$ and by (3.59) we have $\alpha \equiv -1$. Thus, by (3.59) again, we get $(a_1, a_2, a_3, a_4) \equiv -1$.

Set

$$A := \begin{pmatrix} \tilde{a}_{10} & \tilde{a}_{11} \\ \tilde{a}_{20} & \tilde{a}_{21} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{10} \\ a_{21} & -a_{20} \end{pmatrix}, \qquad B := \begin{pmatrix} -\tilde{a}_{10} & -\tilde{a}_{11} \\ \tilde{a}_{20} & \tilde{a}_{21} \end{pmatrix} = \begin{pmatrix} -a_{11} & a_{10} \\ a_{21} & -a_{20} \end{pmatrix}.$$

By $\alpha \equiv -1$ and (3.58), we have $\frac{(f,\tilde{a}_1)}{(f,\tilde{a}_2)} \equiv -\frac{(g,\tilde{a}_1)}{(g,\tilde{a}_2)}$. This implies that $A(f) \equiv B(g)$. Then $g \equiv S(f)$, where $S := B^{-1} \circ A$. By (3.59) and $\alpha \equiv -1$, we have $A(a_4) \equiv B(a_3)$, and $A(a_3) \equiv B(a_4)$. Hence, $S(a_4) = a_3$ and $S(a_3) = a_4$. It is clear that $A(a_1) \equiv B(a_1)$ and $A(a_2) \equiv B(a_2)$. Therefore, $S(a_1) \equiv a_1$, $S(a_2) \equiv a_2$.

We have completed the proof of Theorem 3.1. □

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